3D-PRINTING-TECHNOLOGY IN MATHEMATICS EDUCATION
- EXAMPLES FROM THE CALCULUS

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Abstract: In the last few years, the 3D-printing technology became socially as well as in education increasingly important. The individual production of three-dimensional objects offers a lot of possibilities in teaching and learning mathematics. Materials can be reproduced, developed by the teacher or developed by the students themselves in the mathematics classroom. The field of calculus seems to be particularly suitable for the use of 3D-printing. The opportunities in this domain will be shown by three explicitly discussed examples.

Keywords: 3D-printing, Calculus, Functions, Integraph, Technology.

1. INTRODUCTION

Digital technologies including the 3D-printing technology became more and more important in education in the last years. 3D-printing is an additive manufacturing method used to build a three-dimensional object layer by layer out of liquid plastic. The foundation is a digital model constructed with CAD-Software (cf. Fastermann, 2016). The following article discusses the use of the 3D-printing technology in mathematics education and describes examples in the field of calculus (cf. Dilling, 2019; Witzke & Dilling, 2018).

2. LITERATURE REVIEW

Witzke & Hoffart (2018) distinguish three options for using the 3D-printing technology in the classroom:

The technology can be used to reproduce existing material; The teacher can develop individual material for using in mathematics lessons and the students can comprehend the developmental process; The students can develop 3D-printed objects on their own in the mathematics classroom.

The three scenarios are connected with theories of using material and using technology in mathematics education.

Material is something that can be touched and can be held in the students’ hand (cf. Lengnink, Meyer & Siebel, 2014). In the following parts, we will focus on material students can operate with and that represents a mathematical concept. This material cannot only represent mathematical objects themselves but also relations between the objects. When students operate with the material and interpret it, it can function as a paradigmatic example for the development of relational concepts (cf. Malle, 1984). The use of material may also lead to learning difficulties, especially when students develop misconceptions that obstruct deeper insights (cf. Hanisch, 1985), so it has to be dealt properly and reflectively with the material.

3D-printed material has some advantages over common material. It can be adapted to the needs of the students, so it is very individual. The material can be reproduced as often as needed, so that it enables the working in different classroom formats. Furthermore, the students can be integrated in the process of development.

To understand the use of 3D-printing technology in the mathematics classroom, it is sensible to consider the theory of instrumental genesis. The instrumental genesis describes the process in which an artefact, in this case the CAD-Software and the 3D-printer, becomes an instrument that can be used for problem-solving. The process has two directions. The instrumentation is the development of mental models of the opportunities and limits of the artifact the instrumentalization is the adaption of the artefact to the user’s requirements (cf. Schmidt-Thieme & Weigand, 2015).

CAD-Software is presented in a virtual action space that is a parallel projection of a three-dimensional coordinate system in the Euclidean space, and in which simulations of physical actions are only possible to some degree due to the input devices of the computer (cf. Schumann, 2007). Hartmann, Naf & Reichert (2007) describe this form of representation based on the representational stages of Bruner (1971) as “virtual-enactive”. The transition between the different stages of representation in the 3D-printing process can lead to a deeper understanding of the mathematical content. Therefore, a connection between working with CAD-
Software and the support of spatial imagination as well as mathematical concept building is supposed. Thus, CAD-software as a digital tool can make a useful contribution to the learning of mathematics.

3. METHODS AND RESULTS

The calculus education at school is characterized by tensions between logical precision and elementary plausibility (Vom Hofe, Lotz & Salle, 2015) as well as between algorithm and concept building (Barzel, Fröhlich & Stachniss-Carp, 2004). Winter (1997) stresses that logical rigor is only possible by training the handling with visualizations. Nevertheless, physical models are rarely used in calculus education (Dexheimer, 2014). The 3D-printing technology enables the realization of a number of individual material. Some examples are presented in the following subsections.

3.1. The Software “Graphendrucker”

The software “Graphendrucker” is programmed with the script-based CAD-software OpenSCAD. It enables the user to generate three-dimensional representations of graphs of nearly every function with one variable. Therefore, only the equation of the function and the interval has to be entered. Then the software composes the model of cylinders strung together along the spline. Additional elements can be inserted, e.g. a second graph, small coordinate axis or a whole coordinate system. A screenshot of the interface of the software “Graphendrucker” is shown in Figure 1. The 3D-printed graphs of the sine function and a polynomial of grade 4 can be seen in Figure 2 and 3.

![Figure 1. Screenshot of the interface of the software “Graphendrucker”](image1)

![Figure 2. 3D-printed graph of the sine function](image2)

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\text{In general, four forms of representation of functions (equation, table, graph and verbal description) are distinguished. When working with the software “Graphendrucker”, students use several of those forms. They enter the equation of the function and the software generates a virtual model of the 3D-printable graph. The 3D-printed model has several similarities with the graph of a function, but there are also many differences, e.g. it is touchable and possible to work enactively with it. Further research has to show if the 3D-printed graph is a modified graphical representation or even a quite new form of representation.}

According to Vollrath (1989), three aspects of functional thinking exist. The aspect of assignment is the characteristic that exactly one \( f(x) \) is assigned to each \( x \). Due to the definition of a function, this aspect is highly developed in most students’ thinking. The aspect of covariation focuses on the change of \( f(x) \) when changing \( x \). It is in direct relation to the rate of change (Malle, 2000) and very important for curve sketching (Hahn, 2005). According to Malle (2000), there is a deficit in the students thinking regarding the aspect of covariation. This aspect can be accentuated by sliding along the model with the finger. By not using a coordinate system, the assignment-aspect is faded out so that the covariation is additionally emphasized. The object-aspect refers to the functional correlation as a whole. Characteristics like monotonicity or symmetry are easy to recognize in graphical representations. This aspect is foregrounded additionally by the model as a physical object with which one can work enactively.

The “embodied approach” to the calculus by David Tall (2013) focuses on mental actions with a graph “which we can trace with our fingers and see as an object” (p.299). By sliding along the graph with the fingertip, it is possible to build first conceptions of continuity as “dynamic continuity”. Furthermore, students can feel the slope of the graph with the graph.

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\text{\( f(x) = 0.03 x^4 - 0.3 x^2 \) with a coordinate system}
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extended hand (“to let slope of the hand follow the changing slope”, p.300). Also local linearity can be visualized by covering parts of the graph. With those actions, a conceptual foundation for a later formalism could be built.

Moreover, the 3D-models can be used to renew the classic curve sketching. This is criticized frequently since the use of graphing calculators in German high schools because the graphic representation is permanent available and not anymore a legitimated goal. Instead, Hahn (2005) suggests a belief-orientated curve discussion. The beliefs of the students become uncovered while they communicate about a qualitative given function, e.g. a graph. Thus the 3D-models can be used to deal with central characteristics of functions.

3.2. The Integraph

The integraph is an instrument that mechanically draws the graph of the primitive integral of a graphically given function. With the integraph it is possible to determine indefinite integrals graphically. A 3D-printed integraph can be seen in Figure 4.

![Figure 4. A 3D-printed integraph](image)

First concepts of an integraph trace back to Leibniz (1693). The first actually operating integraph originates from Abdank-Abakanowicz in 1882 and was developed for using in engineering (cf. Willers, 1951). Due to the development of computers, integragraphs are not used anymore by engineers. The model presented in this article was especially developed for using in education, because the mechanism is kept simple and it is possible to draw graphs in the exercise book with it.

The principle of the integraph is as follows. The f(x)-value of the given graph becomes assigned to a directional ruler by a rhomboid-construction. When moving the integraph, the directional ruler draws a continuous line. Because of the equality of the f(x)-values and the slopes this line is the graph of the primitive integral though, it is displaced by one unity. A graphically integrated function can be seen in Figure 5.

![Figure 5. A graphically integrated piecewise linear function](image)

The integraph visualizes the first part of the fundamental theorem of calculus. Blum (1982) developed a new proof of the theorem based on the integraph. Therefore, he reworded the theorem to: “The drawn graph of the primitive integral is the function, that describes the area under the graph of the given function.” The existence of the primitive integral and the area describing function is assumed. The given function can be approximated arbitrarily close by a step function so that in this case the accordance of the two functions can be verified with elementary geometry. Because of the accordance of all approximating functions, there is an accordance of the approximated function to.

This new proof does not require the integraph, but it is a helpful tool that illustrates the idea of the proof and that allows the students to retrace each of the steps. Advantages of the proof are also the recourse to elementary geometry and the opportunity of inserting non-distorting simplifications (cf. Blum, 1982).

A further application of the integraph in the classroom is the graphical differentiation and integration. This is the qualitative drawing of the graph of the derivative or a primitive integral by graphical determination of the derivative or the integral at single points. By using the integraph the function-character and the continuity of the primitive integral become accentuated.

Furthermore, students can use the integraph in the field of technology, for example to determine the area and the moment of a closed curve, or in algebra, for example to determine the zeros of polynomials or to solve algebraic equations.

According to Blum (1982), the use of an integraph in mathematics classroom constitutes an active learning that leads to a rising motivation and challenge of the students. The principle of the integraph can be uncovered by the students because it is no black-box. This results in a deeper understanding of the fundamental theorem and the calculus in general. A
further benefit arises in the occupation with the history and the practical importance of integration.

Today, integraphs are rarely available so that it is difficult to realize the use in mathematics lessons. Hence, Blum (1982) proposes the use of films instead of the integraph itself. The problem of this approach is the inactivity of the students when watching a film. Elschenbroich (2016) develops an applet for the software GeoGebra that represents an integraph. This form of realization does not lead to a really enactive working of the students and the applet is furthermore a black-box. The 3D-printing technology permits the production of an integraph for every student so that they can work enactively and individually with it.

3.3. Functions of two variables

The 3D-printing-technology facilitates the fabrication of models of the graphs of functions of two variables. The OpenSCAD-based software “Graphendrucker mit zwei Variablen” allows the students to generate three-dimensional representations of those graphs only by entering the equation of the function and the intervals for the two variables. By changing the resolution of the model (number of Elements in the single intervals), also discrete functions can be realized. A Screenshot of the software interface is shown in Figure 6, examples for 3D-printed models can be seen in Figure 7 and 8.

Figure 6. Screenshot of the interface of the software “Graphendrucker mit zwei Variablen”

Figure 7. 3D-printed graph of the function \( f(x, y) = 0.5x^2y \) with discrete y-values

Functions of two variables are not part of the mathematics curriculum in German high schools. Nevertheless, many functions students operate with at school are functions of two variables, for example formulas for surfaces and volumes in geometry, arrays of functions, terms in algebra, etc. (Weigand & Flachsmeyer, 1997). Therefore, it is not a new topic which arises the quantity of subject matter but rather a broader perspective on an already taught topic (Kirsch, 1986).

An extension of the concept of functions is furthermore legitimized by didactical deliberations. Functions are a fundamental idea of mathematics, so students should know about diverse facets on this concept. Moreover, the handling with functions of two variables supports spatial imagination and the topic functions as a bridge between mathematics and science as well as between the mathematical subdomains themselves (Weigand & Flachsmeyer, 1997).

Weigand & Flachsmeyer (1997) distinguish five approaches to functions of two variables: discrete functions in extra-mathematical contexts, step functions, arrays of functions, recursively defined sequences with different start values and linear optimization. All those approaches can be supported by 3D-printed models, but in the context of calculus especially arrays of functions are interesting.

The common approaches to arrays of functions in calculus education use a two-dimensional graphical representation and a slide control to change the parameter of the function in dynamic geometry software. This is connected to the disadvantage that the function is only representable with one parameter at one time. By using the three-dimensional printed model, the change of the graph when changing the parameter becomes visible. This change can be also sensed qualitatively by touching the object. When starting to deal with such representations, it can make sense to use discrete functions like in Figure 8 due to the complexity of the continuous representations.
4. DISCUSSION AND CONCLUSION

The theoretical perspectives and the examples from the calculus show the opportunities of the 3D-printing-technology to support mathematics learning but also the challenges that emerge. In addition to the above mentioned applications, further possible applications in the calculus arise in the field of two- and three-dimensional curves, periodic functions, integration, differentiation and solids of revolution. Further research has to show if the anticipated advantages verify in practice.

REFERENCES


